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ON SIMPLE WAVES IN AN ELASTIC-IDEAL PLASTIC MEDIUM*

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Elastic-plastic flow is described by a non-linear hyperbolic /1/ system of equations and an inequality (non-negativity of the factor in the associated law) that is the loading condition. There are overturning waves among the simple waves (SW) of this system of equations. However, only those SW that are not overturning satisfy the loading condition. This fact is known for the case when Hooke's law for elastic deformation and the Mises flow criterion /2/ are assumed; its foundation substantially utilized these particular properties. The absence of SW overturning is established below for a body with an arbitrary smooth flow surface and linear anisotropic elasticity. In this case jumps do not occur from the elastic-plastic SW, which indicates the possibility of solving the problem of the decay of an arbitrary discontinuity without the insertion of jumps of any new kinds that require the search for additional conditions.

Formulation of the problem and of the result. Simple waves, solutions of the form $\mathbf{u}(\theta(x, t))$, are considered for the system of quasilinear equations

$$A(\mathbf{u})\, \partial\mathbf{u}/\partial t + B(\mathbf{u})\, \partial\mathbf{u}/\partial x = 0$$

(A,B are matrices, and $\mathbf{u}$ is the vector of the unknowns). The necessary and sufficient condition for the existence of SW is that the characteristic equation

$$\det(-A(\mathbf{u})\, c + B(\mathbf{u})) = 0 \tag{1}$$

should have a real root $c(\mathbf{u})$. If $c(\mathbf{u})$ is such a non-multiple root and $\mathbf{f}(\mathbf{u})$ is a non-zero solution of the system $(-Ac + B)\,\mathbf{f} = 0$, then a simple wave is found as the solution of the equations

$$d\mathbf{u}/d\theta = \mathbf{f}(\mathbf{u}(\theta)),\ \partial\theta/\partial t = -c(\mathbf{u}(\theta))\, \partial\theta/\partial x \tag{2}$$

The first group of these relationships can, in principle, be integrated. Let $\mathbf{u}(\theta, u_0)$ be its solution ($\mathbf{u}_0$ is a constant vector). Then the change of all the quantities $\mathbf{u}$ in the SW is determined by the evolution of the parameter $\theta(x, t)$. In conformity with the last of Eqs.(2), the initial value $\theta(x, t_0) = \theta_0(x)$ goes over into a constant velocity $c(\mathbf{u}(\theta_0, \mathbf{u}_0))$ in time.

If the derivative $dc/d\theta = (\partial c/\partial \mathbf{u})\, \mathbf{f}(\mathbf{u}(\theta)$ is non-zero, then unlimited growth of derivatives of the solution can occur in the SW (overturning of the SW). The inequality

$$(dc/d\theta)\, \partial\theta/\partial x < 0 \tag{3}$$

is the overturning criterion for SW propagating to the right ($c \geqslant 0$).

Therefore, for $dc/d\theta \neq 0$ the SW of the system of equations under consideration with either growing or decreasing initial profile $\theta_0(x)$ are certainly overturned. The situation is otherwise with elastic-plastic SW. Besides the system of equations they must satisfy the inequality, the loading condition. It it is not satisfied for a certain SW of the system of plastic flow equations, then this SW is not a solution of the elastic-plastic problem. In that case the initial data evolve in conformity with the equations of elasticity theory (unloading occurs).

The loading condition is for the selection of SW of the system of plastic flow equations that have a mechanical meaning. In the case of an isotropic body with a Mises flow surface the inequality (3) is not satisfied for SW selected using it /2/. In other words, in this case the elastic-plastic SW do not overturn. It is clarified below that for the same reason there is no overturning of elastic-plastic SW for a considerably broader class of bodies also.

An elastic-ideal plastic body is considered with an arbitrary smooth flow surface $F(\sigma_{ij}) = 0$ and its associated flow law. As usual the function F is assumed to be convex; the domain of elastic behaviour in the space of the stresses σ_{ij} is determined by the inequality $F(\sigma_{ij}) < 0$, $F(0) < 0$. The law $\sigma_{ij} = A_{ijkl}\varepsilon_{kl}{}^e$, $\varepsilon_{ij}{}^e = B_{ijkl}\sigma_{kl}$ is taken for the elastic strains. The constant elastic moduli possess the symmetry properties

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$$A_{ijkl} = A_{jikl} = A_{ijlk} = A_{klij} \tag{4}$$

The elastic-plastic flow of such a body is described within the framework of the geometrically linear theory by a system /3, 4/ of equations (x^i are Cartesian coordinates, v_i are velocity components, ρ_0 is the density and $F_{ij} \equiv \partial F/\partial\sigma_{ij}$)

$$\rho_0 \frac{\partial v_i}{\partial t} = \frac{\partial \sigma_{ij}}{\partial x^j}, \quad \frac{1}{2}\left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i}\right) = \frac{\partial \lambda}{\partial t} F_{ij} + B_{ijkl}\frac{\partial \sigma_{ij}}{\partial t}, \quad F(\sigma_{ij}) = 0 \tag{5}$$

and the inequality (loading condition)

$$\partial\lambda/\partial t \geqslant 0 \tag{6}$$

it is later established that plane elastic-plastic SW do not overturn for such a body.

Plane SW. The last of relationships (5) can be replaced by the equivalent equation $F_{ij}\,\partial\sigma_{ij}/\partial t = 0$ if the initial data satisfy the condition $F(\sigma_{ij}) = 0$. Using this equation, we find an expression for $\partial\lambda/\partial t$ from the convolution of the second of the relationships (5) with F_{ij}. By using it, system (5), (6) is reduced to the form

$$\rho_0 \partial v_i/\partial t = \partial\sigma_{ij}/\partial x^j, \quad \partial\sigma_{ij}/\partial t = L_{ijkl}e_{kl}, \quad e_{ij} = {}^1/_2\,(\partial v_i/\partial x^j + \partial v_j/\partial x^i)$$
$$\partial\lambda/\partial t = \alpha^{-1}A_{mnpq}e_{mn}F_{pq} \geqslant 0, \quad \alpha = A_{abcd}F_{ab}F_{cd}$$

Here the quantities

$$L_{ijkl} = A_{ijkl} - \alpha^{-1}A_{ijpq}F_{pq}A_{klrs}F_{rs} \tag{7}$$

possess symmetry properties analogous to (4) (all the subscripts run through the values 1, 2, 3).

We obtain the following equation for plane SW of this system, i.e., solutions dependent on $\theta(x, t)$, $x \equiv x^1$, (the prime denotes the derivative with respect to θ)

$$-c\rho_0 v_i' = \sigma_{i1}', \quad -c\sigma_{ij}' = L_{ijk1}v_k' \tag{8}$$
$$-c\lambda' = \alpha^{-1}A_{m1pq}F_{pq}v_m', \quad \partial\theta/\partial t + c\,\partial\theta/\partial x = 0$$

We eliminate the stress from the first equation (by using the second)

$$(-\rho_0 c^2\delta_{ij} + B_{ij})\,v_j' = 0, \quad B_{ij} = L_{i1j1}$$

Together with the remaining relationships of (8) it comprises a system describing elastic-plastic SW. The characteristic Eq.(1) has the following form in the case under consideration:

$$\det(-\rho_0 c^2 I + B) = 0$$

Let us mention one relationship utilized later to investigate SW overturning.

Derivatives of the eigennumber of a symmetric matrix with respect to its components. We consider a symmetric matrix with the components a_{ij}. Let λ_k be its eigenvalues, $\mathbf{w}_k$ the orthonormalized set of its eigenvectors, and w_{kl} $(l = 1, 2, 3)$ the components of the vector $\mathbf{w}_k$ (the subscripts run through the values 1, 2, 3).

Let f be a function of the symmetric matrix a. Its derivative with respect to a is the symmetric matrix $f_{ij}'(a)$ in the expression of the principal linear part of the increment

$$f(a + h) - f(a) = f_{ij}'(a)\,h_{ij} + o(h)$$

(h is an arbitrary symmetric matrix). If the function f is continued in the set of all matrices by the relationship

$$f(a) = f(a/2 + a^T/2)$$

where the superscript T denotes the transpose, then the following equality holds:

$$\frac{\partial f}{\partial a_{ij}}(a) = f_{ij}'\left(\frac{a + a^T}{2}\right)$$

In the case of the symmetric matrix a the right-hand side obviously equals $f_{ij}'(a)$. In this connection, the derivative $f_{ij}'(a)$ is later denoted by $\partial f/\partial a_{ij}$.

The following formula holds (here and in the next two relationships there is no summation over k!):

$$\partial\lambda_k/\partial a_{ij} = w_{ki}w_{kj} \tag{9}$$

It is obtained by differentiating the equality $\lambda_k = w_{km}a_{mn}w_{kn}$ taking the relationship

$$(\partial w_{km}/\partial a_{ij})\,a_{mn}w_{kn} + w_{km}a_{mn}\partial w_{kn}/\partial a_{ij} = \lambda_k\,(\partial/\partial a_{ij})\,(w_{kn}w_{kn}) = 0$$

into account.

Absence of SW overturning. For waves propagating to the right $(c \geqslant 0)$, the necessary and sufficient condition for overturning (3) can be represented in the form

$$d\rho_0 c^2/d\theta \cdot \partial\theta/\partial x < 0 \tag{10}$$

We will show that it is sometimes not satisfied for the elastic-plastic SW under consideration. We will find the first factor as the derivative of the complex function

$$\frac{d\rho_0 c^2}{d\theta} = \frac{\partial\rho_0 c^2}{\partial B_{ij}} \frac{\partial B_{ij}}{\partial F_{kl}} \frac{\partial F_{kl}}{\partial\sigma_{mn}} \sigma_{mn}' \tag{11}$$

Here $B_{ij} = L_{i1j1}$ are components of a symmetric matrix. Its derivative with respect to F_{kl} is evaluated taking (7) into account, while the derivative of its eigenvalue $\rho_0 c^2$ with respect to B_{ij} is evaluated by means of (9) using the eigenvector $v'/|v'|$. We therefore obtain the expression

$$\frac{\partial\rho_0 c^2}{\partial B_{ij}} \frac{\partial B_{ij}}{\partial F_{kl}} = \frac{1}{|v'|^2} v_i' v_j' \frac{\partial}{\partial F_{kl}} (\alpha^{-1} A_{i1rq} F_{pq} A_{j1rs} F_{rs}) =$$
$$-\frac{2}{\alpha |v'|^2} A_{i1pq} F_{pq} v_i' L_{j1kl} v_j' = -\frac{2c^2}{|v'|^2} \lambda' \sigma_{kl}'$$

for their convolution.

The last equality is obtained by using the relationships (8). Substituting it into (11) we find

$$\frac{d\rho_0 c^2}{d\theta} = -2c^2\lambda' H, \quad H = \sigma_{kl}' \frac{\partial^2 F}{\partial\sigma_{kl}\partial\sigma_{mn}} \sigma_{mn}'$$

and the expression on the left-hand side of (10) equals $2cH\lambda'\partial\theta/\partial t = 2cH\partial\lambda/\partial t$.

By virtue of the convexity of the function F, the quantity H is non-negative while the inequality $\partial\lambda/\partial t \geqslant 0$ is the loading condition satisfied during plastic flow. Therefore, Condition (10) is not satisfied and, hence, the elastic-plastic SW for the class of bodies under consideration do not overturn.

REFERENCES

1. MANDEL J., Ondes plastiques dan un milieu indéfini á troites dimensions, J. Mech., 1, 1, 1962; Mekhanika, 5, 1963.
2. KAMENYARZH YA.A., On simple waves and the decay of a discontinuity in an elastic-plastic medium with a Mises condition, PMM, 36, 2, 1972.
3. SEDOV L.I., Mechanics of a Continuous Medium, 2, Nauka, Moscow, 1976.
4. KLYUSHNIKOV V.D., Mathematical Theory of Plasticity, Izd. Moskov. Gos. Univ., Moscow, 1979.

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